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An expansion theorem for water-wave potentials

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Summary

Consider an infinitely long, horizontal cylinder of arbitrary cross section, floating on the free surface of an inviscid, incompressible fluid of infinite depth. The fluid motion is assumed two-dimensional, irrotational and of small amplitude, and it is described by a wave potential satisfying the Laplace equation, the usual linearized free-surface and body-boundary conditions, as well as proper conditions at infinity. A general multipole expansion for the wave potential is derived, converging throughout the fluid domain. Conditions are also stated under which the corresponding expansion for the fluid velocity converges up to and on the body boundary. In this case the multipole expansion may be used in the numerical solution or in the theoretical study of various water-wave problems.

To obtain the expansion, a decomposition of the wave potential in a regular wave, a wave source, a wave dipole and a regular wave-free part is first invoked. Subsequently, using Texeira's series and the conformal mapping between the semicircular region $|\zeta| \geqslant 1$, $\operatorname{Im} \zeta \leqslant 0$, and the fluid domain, it is shown that the regular part of the wave potential can be represented by a convergent series of wave-free multipoles, which are given explicitly in terms of the mapping function.

1. Introduction

Consider an infinitely long, horizontal cylinder floating on the free surface of an inviscid, incompressible fluid under gravity. The fluid is assumed infinitely deep and its motion time-harmonic, two-dimensional, irrotational and of small amplitude. Then, the fluid motion can be described by a velocity potential, called also a wave potential, satisfying a linearized boundary-value problem. Such problems have been treated by many authors using several different methods: boundary-integral equations, multipole expansions, variational principles, hybrid methods (see Wehausen [1,2], Euvrard et al. [3], Mei [4], Euvrard [5] for pertinent surveys). Besides, modern treatments using functional-analytic techniques have recently appeared (see, e.g., Beale [6] and Lenoir [7]).

The method of multipole expansion was initiated by Ursell in 1949, [8,9], and has been widely used thenceforth in the study of various water-wave problems. A rigorous justification of the form that the expansion takes in the case of a semicircular boundary, has been given by Ursell [10]. His arguments were closely related to those developed previously by himself in connection with a uniqueness theorem for the wave potential around a fixed submerged circular cylinder [11]. The corresponding expansion for noncircular boundaries with a vertical axis of symmetry has been given by Ursell [9] and used extensively in the study of radiation problems. In spite of its long-standing and successful use, this expansion is still lacking a rigorous justification, analogous to that given by Ursell for a semicircular boundary.

In the present study a multipole expansion is established for the wave potential outside a floating cylinder of arbitrary cross section. This expansion is used by Athanassoulis [12] to study the existence and uniqueness questions for the radiation problem. Furthermore, it may be used to obtain numerical solutions for the radiation and diffraction problems. In fact, very satisfactory results have recently been obtained for all three rigid modes of motion (sway, heave, roll) of several nonsymmetric bodies (see Lyberopoulos [27] and Lyberopoulos, Athanassoulis and Loukakis [28]).

To obtain the multipole expansion we use the fact that any wave potential may be decomposed into a regular wave, a wave source, a wave dipole and a regular wave-free part. Such a decomposition has been proved by Ursell [10,11] under slightly different circumstances but the proof can be easily extended to the present case by similar reasoning; see Athanassoulis [13]. Subsequently, an expansion of the regular wave-free part in terms of simple wave-free multipoles is obtained with the aid of Texeira's series and the conformal mapping of the semicircular region $|\zeta| \geqslant 1$, $\operatorname{Im} \zeta \leqslant 0$, onto the fluid domain. The corresponding mapping function always exists and, for a wide class of body boundaries, has a reasonable boundary behaviour, ensuring the validity of the expansion up to and including the boundary; see Appendix II. However, from the computational point of view, the construction of this mapping function may be a difficult problem for complicated boundaries.

The author is aware of only one pertinent work (Wehausen [2]) concerned with the multipole expansion of the wave potential outside floating cylinders of arbitrary cross section. The relation between the present and Wehausen's work will be discussed in Sec. 4.

2. Formulation of the problem and decomposition of the wave potential

A Cartesian coordinate system Ox_2x_3 is introduced with Ox_2 on the mean free surface, Ox_3 directed vertically upwards, and the origin O inside the floating body (Fig. 1). A point in the (x_2, x_3)-plane is represented by $x = (x_2, x_3)$ or $w = x_2 + \mathrm{i}x_3$, in complex notation. The mean fluid domain D is considered topologically open, i.e. it does not contain its boundary points. The mean positions of the rigid boundary and the free surface are denoted by ∂D_{B} and ∂D_{F}, respectively. A cross in the upper right side of a pointset symbol denotes the symmetric pointset with respect to the x_2-axis. Accordingly, D^+ and $\partial D_{\mathrm{B}}^+$ denote the symmetric images of D and ∂D_{B}, respectively. Furthermore we define

$$D^* = D \cup D^+ \cup \partial D_{\mathrm{F}} \quad \text{and} \quad \partial D_{\mathrm{B}}^* = \partial D_{\mathrm{B}} \cup \partial D_{\mathrm{B}}^+ ,$$

where $\cup$ denotes set-theoretic union. D^* is topologically open and does not contain the point at infinity.

We also introduce the infinite boundaries ∂D_∞ and ∂D_∞^+ with the following meaning:

$$x \in \partial D_\infty (\partial D_\infty^+) \quad \text{means} \quad |x| \to \infty \text{ and } x_3 \leqslant 0 (x_3 \geqslant 0).$$

It should be emphasized that ∂D_∞ and ∂D_∞^+ cannot be identified with the point at infinity, since $|x|$ is taken to approach infinity, x lying in the lower (upper) half-plane only. On the

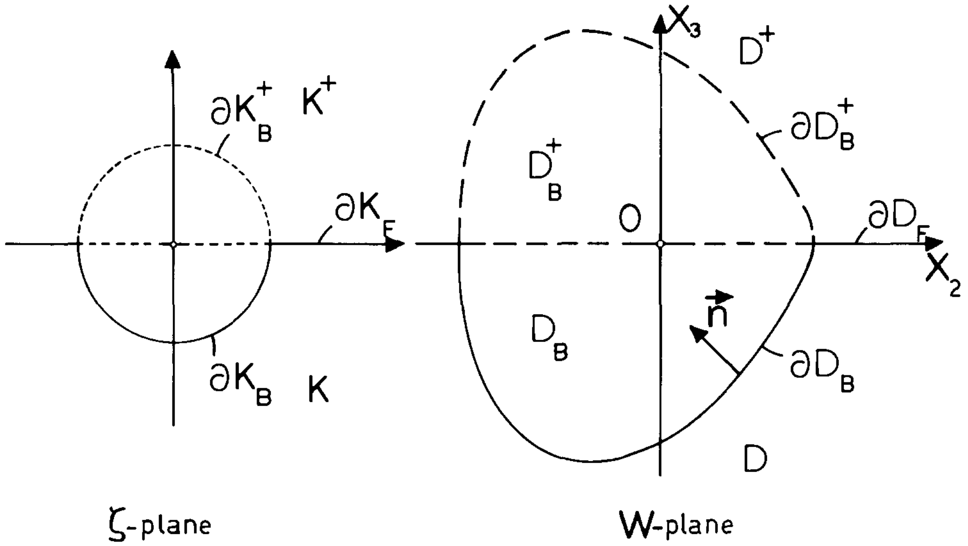

Figure 1. Geometrical description.

contrary, the infinite boundary ∂D^*_∞ of the domain D^*, defined by $\partial D^*_\infty = \partial D_\infty \cup \partial D^+_\infty$, is actually the point at infinity.

We assume that the body boundary ∂D_B performs small-amplitude time-harmonic oscillations with frequency ω and normal velocity $U_n(x, t) = u_{nc}(x) \cos \omega t - u_{ns}(x) \sin \omega t$, $x \in \partial D_B$. Introducing the imaginary unit $j = \sqrt{-1}$, we can write the normal velocity in the form

$$U_n(x, t) = \mathrm{Re}_j\{u_n(x)\, e^{j\omega t}\}, \quad x \in \partial D_B,$$

where $u_n(x) = u_{nc}(x) + j u_{ns}(x)$ is its j-complex amplitude. (1) Then, the fluid motion is described by a velocity potential

$$\Phi(x, t) = \mathrm{Re}_j\{\phi(x)\, e^{j\omega t}\},$$

where $\phi(x)$, the j-complex amplitude of $\Phi(x, t)$, satisfies the Laplace equation

$$\phi,_{22}(x) + \phi,_{33}(x) = 0, \quad x \in D, \tag{2.1}$$

and the boundary conditions

$$k_0\phi(x) - \phi,_3(x) = 0, \quad k_0 = \omega^2/g, \quad x \in \partial D_F, \tag{2.2}$$

$$\frac{\partial\phi(x)}{\partial n} = u_n(x), \quad x \in \partial D_B, \tag{2.3}$$

(1) Two sets of functionally different complex numbers will be used in the present work. The set $\mathbb{C}_j$, of j-complex numbers, and the set $\mathbb{C}_i$, of i-complex numbers. Formal products of i- and j-complex numbers also occur, leading to the new set of ij-complex numbers. The algebraic and topological structure of the latter is developed in Appendix I.

and

$$|\phi,_l(x)| \to 0, \quad l=2,3, \quad x_3 \to -\infty. \tag{2.4}$$

Here $n=(n_2, n_3)$ is the unit normal on ∂D_B directed outwards with respect to the fluid, and $\partial/\partial n$ denotes differentiation along that normal.

A condition at ∂D_∞ is usually added to the above equations. In the case of radiation problems this is the well-known radiation condition, stating that at ∂D_∞ the fluid motion is represented by simple progressive waves, traveling outwards to infinity. In the case of diffraction problems a regular wave component, the incident wave, should be subtracted from the total wave potential, before the radiation condition is applied (John [14]). Nevertheless, in the present work we shall proceed in a different way. We shall seek all possible solutions for the wave potential, which are of slow (polynomial) growth at ∂D_∞ (cf. Ursell [10]):

$$|\phi,_l(x)| < M|x|^N, \quad \begin{array}{l} l=2,3, \quad M>0, \\ N \geqslant 0, \quad x \in \partial D_\infty. \end{array} \tag{2.5}$$

Condition (2.5), though weaker than the more natural boundedness condition, i.e. $|\phi,_l(x)| < M$, $x \in \partial D_\infty$, eventually leads to the same general form for $\phi(x)$.

Equations (2.1) to (2.5) define the water-wave problem that is to be studied in the present work.

We now introduce the ij-complex wave potential

$$F(w) = \phi(x) + \mathrm{i}\psi(x), \tag{2.6}$$

where $\phi(x)$ and $\psi(x)$ are the j-complex amplitudes of the velocity potential and the corresponding stream function, respectively. Equations (2.1) to (2.5) are then transformed into the following:

$$F(w) \text{ is i-analytic}, \quad w \in D, \tag{2.7}$$

$$\mathrm{Im}_\mathrm{i}\left\{\frac{\mathrm{d}F(w)}{\mathrm{d}w} + \mathrm{i}k_0 F(w)\right\} = 0, \quad w \in \partial D_\mathrm{F}, \tag{2.8}$$

$$\mathrm{Re}_\mathrm{i}\left\{n(w)\frac{\mathrm{d}F(w)}{\mathrm{d}w}\right\} = u_\mathrm{n}(w), \quad w \in \partial D_\mathrm{B}, \tag{2.9}$$

$$\left|\frac{\mathrm{d}F(w)}{\mathrm{d}w}\right|_{\mathrm{C}_{\mathrm{ij}}} \to 0, \quad x_3 \to -\infty, \tag{2.10}$$

$$\left|\frac{\mathrm{d}F(w)}{\mathrm{d}w}\right|_{\mathrm{C}_{\mathrm{ij}}} < M|w|^N, \quad M>0, \quad N \geqslant 0, \quad w \in \partial D_\infty, \tag{2.11}$$

respectively. Here $|\cdot|_{\mathrm{C}_{\mathrm{ij}}}$ is the absolute value of ij-complex numbers (see Appendix I) and $n(w) = n_2 + \mathrm{i}n_3$ is the unit normal on ∂D_B considered as an i-complex number.

Equations (2.7) to (2.11) constitute an alternative formulation of the examined water-wave problem. This formulation is particularly suitable for the study of our problem in the context of analytic-function theory.

Using the well-known reduction method and proceeding along lines due to Ursell [10], [11], we arrive at

THE DECOMPOSITION THEOREM (Athanassoulis [13]): *Any function* $F(w)$ *satisfying the conditions* (2.7), (2.8), (2.10) *and* (2.11) *can be decomposed in the form*

$$F(w) = B\,\mathrm{e}^{-\mathrm{i}k_0 w} + S_1 F_1(w) + S_2 F_2(w) + \theta(w), \quad w \in D, \tag{2.12}$$

where $B \in \mathbb{C}_{\mathrm{ij}}$, S_1, $S_2 \in \mathbb{C}_{\mathrm{j}}$ *(see Appendix I for the definition of* $\mathbb{C}_{\mathrm{ij}}$*);* $F_1(w)$, $F_2(w)$ *are wave singularities at the origin defined by*

$$F_m(w) = \mathrm{e}^{-\mathrm{i}k_0 w} \int_{\infty + \mathrm{i}0}^{w} u^{-m}\,\mathrm{e}^{\mathrm{i}k_0 u}\mathrm{d}u, \quad m = 1, 2, \quad w \in D; \tag{2.13}$$

$\theta(w)$ *is an* ij*-complex function regular in* $D^*_\infty = D^* \cup \{\infty\}$, *satisfying the free-surface condition*

$$\mathrm{Im}_{\mathrm{i}}\left\{\frac{\mathrm{d}\theta(w)}{\mathrm{d}w} + \mathrm{i}k_0\theta(w)\right\} = 0, \quad w \in \partial D_{\mathrm{F}}. \tag{2.14}$$

That is, any wave potential $F(w)$ *can be expressed as the sum of a regular wave, a wave source, a wave dipole, and a wave-free part regular throughout* D^*_∞. (2)

REMARK: The use of $F_1(w)$ and $F_2(w)$ is not obligatory for the decomposition (2.12). Higher-order singularities, defined by (2.13) with $m > 2$, may also be used. Actually, what is needed is a pair of odd- and even-order wave singularities.

3. The multipole expansion of the wave potential

To obtain the multipole expansion of the wave potential an explicit representation is needed for the regular wave-free function $\theta(w)$, appearing in (2.12). Since the domain of regularity D^*_∞ of this function is not annular, an ordinary Laurent series in the w-plane cannot be used to represent it throughout D^*_∞. Nevertheless, one may always use a Texeira series representation (see, e.g., Whittaker and Watson [15], §7.31 or Sansone and Gerretsen [16], §3.13) of the form

$$\theta(w) = \sum_{n=0}^{\infty} b_n\{g(w)\}^{-n}, \tag{3.1}$$

where $g(w)$ is the function mapping conformally D^* onto the exterior of the unit circle. This representation, although completely general, is not convenient for our purpose, since the restrictions induced on b_n's by the free-surface condition (2.14) cannot be obtained in a simple manner. This difficulty may be, however, surmounted by making the change of variable $w = f(\zeta)$, where $f(\zeta)$ is the function mapping conformally the exterior of the unit circle in the ζ-plane onto the domain D^* in the w-plane, i.e. the inverse function of $g(w)$.

(2) The usual wave source and wave dipole can be easily obtained by linear combinations of $F_1(w)$ and $F_2(w)$.

The function $f(\zeta)$ is represented by a Laurent series of the form

$$f(\zeta)=\sum_{l=1}^{\infty} C_l \zeta^{2-l}, \quad C_l \in \mathbb{R}, \quad |\zeta|>1, \tag{3.2}$$

converging uniformly and absolutely in any compact subregion of $|\zeta|>1$. Furthermore, under additional conditions, the series (3.2) converges uniformly and absolutely on any region $\{\zeta;\ 1 \leqslant |\zeta| \leqslant R\}$, $R>1$. These conditions, as well as other useful properties of the mapping function $f(\zeta)$, are reported in Appendix II.

Let us now introduce the notation in the ζ-plane (the transformed plane). A point in it is represented by $\zeta=\xi_2+\mathrm{i}\xi_3$, $\xi_2, \xi_3 \in \mathbb{R}$. The domains K, K^+ and K^*, corresponding to D, D^+ and D^*, respectively (Fig. 1), are defined by

$$K(K^+)=\{\zeta;\ 1<|\zeta|<\infty, \xi_3<0(\xi_3>0)\}, \quad K^*=K \cup K^+ \cup \partial K_{\mathrm{F}},$$

where

$$\partial K_{\mathrm{F}}=\{\zeta;\ |\xi_2|>1, \xi_3=0\},$$

is the inverse image of the free surface ∂D_{F}. The boundaries ∂K_{B}, $\partial K_{\mathrm{B}}^+$ and $\partial K_{\mathrm{B}}^*$, corresponding to ∂D_{B}, $\partial D_{\mathrm{B}}^+$ and $\partial D_{\mathrm{B}}^*$, respectively, are defined by

$$\partial K_{\mathrm{B}}(\partial K_{\mathrm{B}}^+)=\{\zeta;\ |\zeta|=1, \xi_3 \leqslant 0(\xi_3 \geqslant 0)\}, \quad \partial K_{\mathrm{B}}^*=\partial K_{\mathrm{B}} \cup \partial K_{\mathrm{B}}^+.$$

Finally, the symbol K_∞^* is used to denote the domain K^* including the point at infinity, i.e. $K_\infty^*=K^* \cup \{\infty\}$.

Introducing the change of variable $w=f(\zeta)$, $\zeta=g(w)$, into (3.1) we obtain the following lemma, which is of essential importance for our further considerations.

LEMMA 1: *Any function $\theta(w)$ that is regular in D_∞^* and satisfies the condition* (2.14) *may be represented in the form*

$$\theta(w)=\theta_1(g(w)), \tag{3.3}$$

where the function $\theta_1(\zeta)$ is regular in K_∞^* and satisfies the condition

$$\operatorname{Im}_{\mathrm{i}}\left\{\frac{\mathrm{d}\theta_1(\zeta)}{\mathrm{d}\zeta}+\mathrm{i}k_0 \frac{\mathrm{d}f(\zeta)}{\mathrm{d}\zeta}\theta_1(\zeta)\right\}=0, \quad \zeta \in \partial K_{\mathrm{F}}. \tag{3.4}$$

Lemma 1 reduces the representation of the regular wave-free potential $\theta(w)$ to the representation of a function $\theta_1(\zeta)$, regular in the annular domain K_∞^* and satisfying the modified free-surface condition (3.4). Let us now find the general form of such a function. At first, we have

$$\theta_1(\zeta)=\sum_{n=1}^{\infty} B_n \zeta^{-n}, \quad B_n \in \mathbb{C}_{ij}, \quad \zeta \in K_\infty^*, \tag{3.5}$$

$(B_0=\theta_1(\infty)=\theta(\infty)=0)$. The series (3.5) converges uniformly and absolutely in any compact subregion of the annulus K^*.

Using now the expansions (3.2) and (3.5) we find

$$\left\{\frac{\mathrm{d}}{\mathrm{d}\zeta}+\mathrm{i}k_0\frac{\mathrm{d}f(\zeta)}{\mathrm{d}\zeta}\right\}\sum_{n=1}^{\infty}B_n\zeta^{-n}$$

$$=\mathrm{i}k_0C_1B_1\zeta^{-1}+\sum_{n=1}^{\infty}\left\{-nB_n+\mathrm{i}k_0\sum_{l=1}^{n+1}(2-l)C_lB_{n+2-l}\right\}\zeta^{-(n+1)}. \tag{3.6}$$

Substituting (3.6) into (3.4) and setting

$$B_n=B_n'+\mathrm{i}B_n'',\quad B_n',\,B_n''\in\mathbb{C}_\mathrm{j},$$

we obtain

$$k_0C_1B_1'\xi_2^{-1}+\sum_{n=1}^{\infty}\left\{-nB_n''+k_0\sum_{l=1}^{n+1}(2-l)C_lB_{n+2-l}'\right\}\xi_2^{-(n+1)}=0,\quad |\xi_2|>1,$$

which implies

$$B_1'=0,\quad nB_n''=k_0\sum_{l=1}^{n+1}(2-l)C_lB_{n+2-l}',\quad n=1,2,3,\dots. \tag{3.7}$$

Inserting the latter relations into the general expansion (3.5) we deduce

$$\theta_1(\zeta)=\sum_{n=1}^{\infty}B_n'\zeta^{-n}+\mathrm{i}k_0\sum_{n=1}^{\infty}\sum_{l=1}^{n+1}(2-l)C_l\frac{B_{n+2-l}'}{n}\zeta^{-n}$$

$$\equiv S_1(\zeta)+\mathrm{i}k_0S_2(\zeta). \tag{3.8}$$

The double series $S_2(\zeta)$ will be now rearranged in a manner revealing the connection of $\theta_1(\zeta)$ with the usual wave-free multipoles for symmetric bodies as well as with the generalized multipoles given by Wehausen [2]:

$$\frac{\mathrm{d}S_2(\zeta)}{\mathrm{d}\zeta}=-\sum_{n=1}^{\infty}\sum_{l=1}^{n+1}(2-l)C_lB_{n+2-l}'\zeta^{-(n+1)}$$

$$=-\zeta^{-1}\sum_{n=2}^{\infty}\sum_{l=1}^{n}(2-l)C_lB_{n+1-l}'\zeta^{1-n}$$

$$=-\sum_{n=2}^{\infty}\sum_{l=1}^{\infty}(2-l)C_lB_n'\zeta^{1-l-n}.$$

Integrating term-by-term and using

$$S_2(\infty)=\theta_1(\infty)=0,$$

we find

$$S_2(\zeta)=-\sum_{n=2}^{\infty}\sum_{l=1}^{\infty}\frac{(2-l)C_lB_n'\zeta^{2-l-n}}{2-l-n}.$$

Upon substituting the above equation into (3.8) we obtain

$$\theta_1(\zeta) = \sum_{n=2}^{\infty} d_n M_n(\zeta), \quad \zeta \in K_{\infty}^*, \tag{3.9}$$

where $d_n \equiv B_n'$ are j-complex constants, and

$$M_n(\zeta) = \zeta^{-n} - \mathrm{i}k_0 \sum_{l=1}^{\infty} \frac{(2-l)C_l \zeta^{2-l-n}}{2-l-n}, \quad n = 2, 3, \ldots \tag{3.10}$$

The functions $M_n(\zeta)$ satisfying the free-surface condition and vanishing at infinity, are the generalized wave-free multipoles. These multipoles are identical with those derived by Wehausen ([2], p. 110), although the final expansion for $F(w)$ obtained there, is not the same as the expansion derived in the present work (see Sec. 4).

When Ox_3 is a vertical axis of symmetry of ∂D_{B}, then $C_{2l} = 0$, $l = 0, 1, 2, \ldots$, and the subsequences

$$M_{2n}(\zeta) = \zeta^{-2n} - \mathrm{i}k_0 \sum_{l=1}^{\infty} \frac{(3-2l)C_{2l-1}\zeta^{3-2l-2n}}{3-2l-2n}$$

and

$$M_{2n+1}(\zeta) = \zeta^{-(2n+1)} - \mathrm{i}k_0 \sum_{l=1}^{\infty} \frac{(3-2l)C_{2l-1}\zeta^{2-2l-2n}}{2-2l-2n}$$

of the generalized wave-free multipoles coincide with the usual symmetric and antisymmetric wave-free multipoles, respectively.

Using the representation (3.9), in conjunction with the decomposition theorem, we arrive at

THE EXPANSION THEOREM: *Suppose that the function $F(w)$ satisfies the conditions* (2.7), (2.8), (2.10) *and* (2.11). *Then, it may be expanded in the form*

$$F(w) = B\,\mathrm{e}^{-\mathrm{i}k_0 w} + S_1 F_1(w) + S_2 F_2(w) + \sum_{n=2}^{\infty} d_n M_n(\zeta), \quad w \in D, \tag{3.11}$$

where B, S_1, S_2, $F_1(w)$ and $F_2(w)$ are defined as in the decomposition theorem, d_n are j-complex constants, $M_n(\zeta)$ are the wave-free multipoles defined by (3.10), $\zeta = g(w) = f^{-1}(w)$, *and $f(\zeta)$ is the function mapping conformally K^* onto D^*.*

The series in (3.11) converges uniformly and absolutely in any compact subregion of K^* and may be differentiated term-by-term any number of times there.

The asymptotic form of the wave potential at ∂D_{∞} may be now easily obtained. From (3.11) it follows that $F(w)$ can be written in the form

$$F(w) = F_{\infty}(w) + F_{\mathrm{R}}(w), \tag{3.12}$$

where

$$F_{\infty}(w) = B^{\pm}\,\mathrm{e}^{-\mathrm{i}k_0 w}, \quad B^{\pm} \in \mathbb{C}_{\mathrm{ij}}, \quad w \in D, \quad x_2 \to \pm\infty, \tag{3.13}$$

$$B^+ = B, \quad B^- = B - 2\pi \mathrm{i} S_1 + 2\pi k_0 S_2, \tag{3.14}$$

and

$$|F_\mathrm{R}(w)|_{\mathbb{C}_{ij}} < \frac{A_2}{|x_2|}, \quad |F_\mathrm{R}(w)|_{\mathbb{C}_{ij}} < \frac{A_3}{|x_3|}, \quad A_2, A_3 > 0, \quad w \in D. \tag{3.15}$$

From (3.13) it is found, by straightforward algebraic manipulations, that the j-complex amplitude $\phi(x)$ of the wave potential has the following asymptotic form at ∂D_∞:

$$\phi(x) = \phi_\infty(x) + \phi_\mathrm{R}(x), \quad x \in D, \tag{3.16}$$

$$\phi_\infty(x) = \frac{g}{\omega} \mathrm{e}^{k_0 x_3} \left\{ h^\pm_\rightarrow \, \mathrm{e}^{-\mathrm{j}(k_0 x_2 + \delta^\pm_\rightarrow)} + h^\pm_\leftarrow \, \mathrm{e}^{\mathrm{j}(k_0 x_2 + \delta^\pm_\leftarrow)} \right\}, \quad x_2 \to \pm\infty, \tag{3.17}$$

$$|\phi_\mathrm{R}(x)| < \frac{A_2}{|x_2|} \quad \text{and} \quad |\phi_R(x)| < \frac{A_3}{|x_3|}, \quad x \in D. \tag{3.18}$$

In the equation (3.17), $h^\pm_\rightarrow$, $h^\pm_\leftarrow$ are four positive constants representing the amplitudes of four simple progressive waves at ∂D_∞, and $\delta^\pm_\rightarrow$, $\delta^\pm_\leftarrow$ are four real constants expressing the phase lags between these waves *. (Actually one of these constants is redundant but we shall not dwell on this point here.) The eight aforementioned constants are expressed in terms of $B^\pm = B^\pm_\mathrm{RR} + \mathrm{j} B^\pm_\mathrm{RI} + \mathrm{i} B^\pm_\mathrm{IR} + \mathrm{ij} B^\pm_\mathrm{II}$ by means of relations

$$h^\pm_\rightarrow = \frac{\omega}{2g} \left\{ \left(B^\pm_\mathrm{RR} - B^\pm_\mathrm{II} \right)^2 + \left(B^\pm_\mathrm{RI} + B^\pm_\mathrm{IR} \right)^2 \right\}^{1/2}, \quad \tan \delta^\pm_\rightarrow = \frac{B^\pm_\mathrm{RI} + B^\pm_\mathrm{IR}}{B^\pm_\mathrm{II} - B^\pm_\mathrm{RR}}, \tag{3.19a}$$

$$h^\pm_\leftarrow = \frac{\omega}{2g} \left\{ \left(B^\pm_\mathrm{RR} + B^\pm_\mathrm{II} \right)^2 + \left(B^\pm_\mathrm{RI} - B^\pm_\mathrm{IR} \right)^2 \right\}^{1/2}, \quad \tan \delta^\pm_\leftarrow = \frac{B^\pm_\mathrm{RI} - B^\pm_\mathrm{IR}}{B^\pm_\mathrm{II} + B^\pm_\mathrm{RR}}. \tag{3.19b}$$

On the basis of (3.17), it is deduced that at either $x_2 \to +\infty$ or $x_2 \to -\infty$, the wave potential may behave as: (i) a simple progressive outgoing wave, or (ii) a simple progressive incoming wave, and/or (iii) a simple stationary wave (cf. Newman [17] and Guével et al. [18]). In this connection it is worthwhile to notice that the a priori use of a radiation condition eliminates the possibilities (ii) and (iii).

Let us now examine more closely the form the expansion takes in the special case of a radiation problem, which is characterized by

$$h^+_\leftarrow = h^-_\rightarrow = 0. \tag{3.20}$$

In virtue of (3.19) it is easily seen that (3.20) is equivalent to

$$B^\pm = \Lambda^\pm (1 \mp \mathrm{ij}), \quad \Lambda^\pm \in \mathbb{C}_\mathrm{j}, \tag{3.21}$$

from which, in conjunction with (3.14), we find

$$B = \Lambda^+(1 - \mathrm{ij}), \quad S_1 = \frac{-\mathrm{j}(\Lambda^+ + \Lambda^-)}{2\pi}, \quad S_2 = \frac{\Lambda^- - \Lambda^+}{2\pi k_0}. \tag{3.22}$$

* The arrows in h's and δ's indicate the direction of the waves.

The expansion (3.11) takes then the form

$$F(w) = \Lambda^+ G^+(w) + \Lambda^- G^-(w) + \sum_{n=2}^{\infty} d_n M_n(\zeta), \tag{3.23}$$

where $\Lambda^{\pm}$ and d_n are j-complex constants and

$$G^+(w) = (1 - \mathrm{ij})\, \mathrm{e}^{-\mathrm{i}k_0 w} - \frac{\mathrm{j}}{2\pi} F_1(w) - \frac{1}{2\pi k_0} F_2(w), \tag{3.24a}$$

$$G^-(w) = \frac{-\mathrm{j}}{2\pi} F_1(w) + \frac{1}{2\pi k_0} F_2(w). \tag{3.24b}$$

The coefficients $\Lambda^{\pm}$ and d_n in the expansion (3.23) have to be determined through the body boundary condition (2.9). It is convenient to rewrite this condition in terms of the transformed variable ζ:

$$\mathrm{Re}_{\mathrm{i}}\left\{ -\zeta \frac{\mathrm{d}F(\zeta)/\mathrm{d}\zeta}{|\mathrm{d}f(\zeta)/\mathrm{d}\zeta|} \right\} = u_n(\theta), \quad \zeta = \mathrm{e}^{\mathrm{i}\theta}, \quad -\pi \leqslant \theta \leqslant 0, \tag{3.25}$$

where $F(\zeta)$ and $u_n(\theta)$ stand for $F(f(\zeta))$ and $u_{\mathrm{n}}(f(\mathrm{e}^{\mathrm{i}\theta}))$, respectively. It should be noted that $\mathrm{d}f(\mathrm{e}^{\mathrm{i}\theta})/\mathrm{d}\zeta$ exists and is different from zero provided that $\partial D_{\mathrm{B}}^*$ is of class $\mathscr{C}^2$ (see Appendix II). Substituting (3.23) into (3.25) and differentiating term-by-term we obtain

$$\mathrm{Re}_{\mathrm{i}}\left[\Lambda^+ H^+(\zeta) + \Lambda^- H^-(\zeta) + \sum_{n=2}^{\infty} n d_n \left\{ \zeta^{-n} + \frac{\mathrm{i}k_0}{n} \sum_{l=1}^{\infty} (2-l) C_l \zeta^{2-l-n} \right\} \right]_{\zeta = \mathrm{e}^{\mathrm{i}\theta}}$$

$$= V(\theta), \quad -\pi \leqslant \theta \leqslant 0, \tag{3.26}$$

where $H^{\pm}(\zeta) = -\zeta \mathrm{d}G^{\pm}(f(\zeta))/\mathrm{d}\zeta$ and $V(\theta) = u_{\mathrm{n}}(\theta)|\mathrm{d}f(\mathrm{e}^{\mathrm{i}\theta})/\mathrm{d}\zeta|$. The termwise differentiation is justified if

$$\sum |(2-l)C_l| < +\infty \quad \text{and} \quad \sum |n d_n| < +\infty. \tag{3.27a, b}$$

Relation (3.27a) is actually a smoothness condition on the body boundary; it is certainly satisfied when $\partial D_{\mathrm{B}}^*$ is of class $\mathscr{C}^2$. Relation (3.27b), which is, in fact, a smoothness condition on the wave potential, may be assumed a priori, provided that it will be checked a posteriori, when the solution of the problem will have been accomplished (cf. Ursell [19] and Athanassoulis [12]).

However, if the left-hand side of (3.26) is interpreted in a limiting sense, as $|\zeta| \to 1^+$, condition (3.27b) may be replaced by the weaker one

$$\sum |n d_n|^2 < +\infty, \tag{3.27c}$$

which ensures the L^2-convergence of the infinite series in (3.26). This viewpoint has been adopted by the author in [12].

4. Discussion

The derived multipole expansion may be useful either in the theoretical or in the numerical study of water-wave radiation and diffraction problems.

The expansion theorem appears to be new in its general form. It is an extension of the corresponding theorem proved by Ursell [10] for fluid regions exterior to a semicircle.

The only previous work known to the author dealing with the application of the multipole expansion method for two-dimensional floating bodies of arbitrary shape is the one by Wehausen [2]. Wehausen has worked in the transformed plane from the outset, taking firstly a representation of the wave potential in terms of an infinite series of wave multipoles (Eqn. 5.44, p. 107). Then, by rearranging this series, he has ended in an expansion (Eqn. 5.57, p. 112) similar to the present one (3.11). However, the second-order wave singularity and the first wave-free multipole are not included in Wehausen's expansion. This discrepancy may be attributed to the rearrangement of the wave-multipoles infinite series.

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Appendix I

The algebraic and topological structure of ij*-complex numbers*

As we saw in Sec. 2, it is convenient to formulate our problem with the aid of two non-interacting imaginary units i and j. The need of using two imaginary units is due to the fact that water-wave problems are dynamical problems in which the field equation is the one of Laplace. So, an imaginary unit j is used to simplify the time-dependence (in the time-harmonic case) while, at the same time, another imaginary unit i is introduced to transform the (x_2, x_3)-plane into a complex plane, and pairs of real harmonic functions into single complex analytic functions. Such a formalism has been introduced in the context of water-wave problems many years ago (see Wehausen and Laitone [20], §11, 19, 21) and has been used may times thenceforward.

The simultaneous presence of the two imaginary units i and j leads to a new kind of numbers, called ij-complex numbers, the set of which is denoted by $\mathbb{C}_{ij}$. In the present appendix we shall develop in a fairly complete manner the algebraic and topological structure of the set $\mathbb{C}_{ij}$. Incidentally, a comparison with the superficially similar hypercomplex numbers (Hamilton's quaternions) will be made.

An element $A \in \mathbb{C}_{ij}$ may be represented by anyone of the following three forms:

$$A = \alpha + \mathrm{i}\beta + \mathrm{j}\gamma + \mathrm{ij}\delta; \quad \mathrm{ij} \equiv \mathrm{ji}; \quad \alpha, \beta, \gamma, \delta \in \mathbb{R}, \tag{I.1}$$

$$A = a + \mathrm{i}b; \quad a, b \in \mathbb{C}_{\mathrm{j}}, \tag{I.2}$$

$$A = z + \mathrm{j}w; \quad z, w \in \mathbb{C}_{\mathrm{i}}. \tag{I.3}$$

The representations (I.1), (I.2) and (I.3) are superficially identical with the representations of quaternions (see, e.g., MacLane and Birkhoff [21], p. 253, and Kostrikin [22], p. 426). The algebraic structure of $\mathbb{C}_{ij}$ is, however, essentially different from that of quaternions, because of the different definition of multiplication.

The following four projection operators may be naturally introduced:

$$\mathrm{Re}_i A = a \in \mathbb{C}_j, \quad \mathrm{Im}_i A = b \in \mathbb{C}_j,$$

$$\mathrm{Re}_j A = z \in \mathbb{C}_i, \quad \mathrm{Im}_j A = w \in \mathbb{C}_i.$$

Let us now define the basic notions and operations in $\mathbb{C}_{ij}$, using the representation (I.3). Let $A = z + jw$, $A_k = z_k + jw_k$, $k = 1, 2$, be in $\mathbb{C}_{ij}$. Then, we define

Equality: $A_1 = A_2 \Leftrightarrow z_1 = z_2$ and $w_1 = w_2$;

Addition: $A_1 + A_2 = (z_1 + z_2) + j(w_1 + w_2)$; (I.4)

Zero element: $0_{\mathbb{C}_{ij}} = 0 + j0$;

Multiplication: $A_1 \cdot A_2 = (z_1 z_2 - w_1 w_2) + j(z_1 w_2 + z_2 w_1)$; (I.5)

Unit element: $1_{\mathbb{C}_{ij}} = 1 + j0$;

Scalar multiplication: $\lambda A = (\lambda z) + j(\lambda w)$, $\lambda \in \mathbb{C}_i$; (I.6)

Absolute value: $|A|_{\mathbb{C}_{ij}} = (|z|^2 + |w|^2)^{1/2}$. (1.7)

The law of multiplication (I.5) is different from the law of multiplication of quaternions. In fact, if $Q_k = z_k + jw_k$, $k = 1, 2$, are two quaternions, their product is defined by

$$Q_1 \cdot Q_2 = (z_1 z_2 - \bar{w}_1 w_2) + j(\bar{z}_1 w_2 + z_2 w_1).$$

The consequences of this difference will be discussed in the sequel.

It is easy to see that the set $\mathbb{C}_{ij}$ equipped with the operations of addition (I.4) and multiplication (I.5) becomes a commutative ring. This commutative ring does not, however, possess the structure of a field, since there exist nonzero elements of it which do not have multiplicative inverses. With regard to this point we state the following two propositions.

PROPOSITION 1: *The multiplicative inverse of the* ij-*complex number* $A = z + jw$ *exists and is uniquely defined by*

$$A^{-1} = \frac{z}{z^2 + w^2} + j\left(-\frac{w}{z^2 + w^2}\right),$$

provided that $z^2 + w^2 \neq 0$. *The set* J *of noninvertible elements has the form*

$$J = \{B;\ B = (\pm i + j)w,\ w \in \mathbb{C}_i\}$$

and forms a proper ideal in $\mathbb{C}_{ij}$.

PROPOSITION 2: *The numbers* $A_1 = (\pm i + j)w$, $A_2 = (\mp i + j)w$, $w \in \mathbb{C}_i$, *are zero divisors in* $\mathbb{C}_{ij}$, *i.e. their product equals to zero though both being different from zero.*

We can now explain the essential algebraic difference between the ij-complex numbers and the quaternions. The former constitute a commutative ring in which there exist nonzero, noninvertible elements and zero divisors; the latter form a noncommutative ring in which all nonzero elements are invertible.

The existence of noninvertible ij-complex numbers does not affect our treatment in the present work, since it is not needed to divide by a general ij-complex number.

The set $\mathbb{C}_{ij}$ equipped with the operations of addition, multiplication and scalar multiplication, and with the norm (I.7), becomes a commutative Banach algebra over the field of i-complex numbers. Various algebraic and topological consequences of this fact can be found in standard treatises on the subject, e.g. in the book by Gaal [23].

Appendix II

On the mapping function $f(\zeta)$

In this appendix we collect, in the form of a theorem, all properties of the mapping function $f(\zeta)$ which have been used in the present work.

THEOREM: *Let the boundary ∂D_B^* be a rectifiable closed Jordan curve, i.e. a simple, continuous, closed curve with finite length. Then the following statements are true:*

(i) There exists a function $f(\zeta)$, analytic in K^* and with a simple pole at infinity, mapping conformally the domain K^* onto the domain D^*. (This is a special case of the Riemann mapping theorem).

(ii) Under the additional condition $\mathrm{d}f(\infty)/\mathrm{d}\zeta > 0$, $f(\zeta)$ is uniquely determined.

(iii) The function $f(\zeta)$ can be extended onto the boundary ∂K_B^*, establishing a one-to-one and bicontinuous correspondence between the points of ∂K_B^* and ∂D_B^* (Osgood-Caratheodory's theorem).

(iv) The function $f(\zeta)$ has a Laurent expansion

$$f(\zeta) = \sum_{l=1}^{\infty} C_l \zeta^{2-l}, \quad C_1 > 0,$$

which converges absolutely and uniformly on the set $K \cup \partial K_B$. Moreover,

$$\sum |C_l| < +\infty.$$

If ∂D_B^* is symmetric with respect to the real axis, all C_l's are real.

Suppose, in addition, that ∂D_B^* is of class $\mathscr{C}^2$, i.e. it is smooth and its tangent vector is continuously differentiable. Then also the following holds:

(v) The mapping function $f(\zeta)$ can be extended as a differentiable function onto the boundary ∂K_B^*. Moreover, its derivative $\mathrm{d}f(\zeta)/\mathrm{d}\zeta$ does not vanish on ∂K_B^* (Kellogg's theorem), and the coefficients C_l satisfy the relation

$$\sum |(2-l)C_l| < +\infty.$$

The proofs of the propositions collected in the above theorem may be found in various treatises concerning the theory of complex functions and conformal mapping. See, e.g., Markushevich [24], ch. 1 and 2, Hille [25], ch. 17, and Tsuji [26], ch. IX, §3, for the proof of Kellogg's theorem.

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